An Extension to Models for Cosmic String Formation

Andrew Yates* and T.W.B. Kibble†
Blackett Laboratory, Imperial College, South Kensington, London SW7 2BZ, U.K.
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The canonical Monte-Carlo algorithm for simulating the production of string-like topological defects at a phase transition is extended by introducing a distribution of domain sizes. A strong correlation is found between the fraction in the form of 'infinite' string and the variance of the volume of the regions of constant phase.

I. INTRODUCTION

Cosmic strings may have formed at a phase transition in the early universe [1,2]. Information about the initial statistics of a string network, after the point at which thermal fluctuations become unimportant and the strings are 'frozen in', has largely emerged from the numerical simulations first performed by Vachaspati and Vilenkin [3]. The simplest case, involving the spontaneous breaking of a $U(1)$ symmetry, is mimicked by assigning a phase between 0 and 2π to each point on a regular lattice. The lattice spacing then corresponds to a correlation length ξ characteristic of the scalar field acquiring the non-zero vacuum expectation value. To look for field configurations with non-trivial topology, the 'geodesic rule' is invoked. This proposes that to minimise gradient energy the field will follow geodesic paths on the vacuum manifold as a path in configuration space is traversed. The phase will thus follow the 'shortest path' between values on adjacent lattice sites. A winding of $\pm 2\pi$ around a plaquette in the lattice means that a line-like distribution of zeroes of the field will pierce it — a cosmic string.

If we impose periodic boundary conditions on our lattice, it becomes obvious that all string must be in the form of closed loops. One would expect that this procedure gives a string configuration in our box that is statistically similar to that when neighbouring, causally disconnected regions are present [4]. From the distribution of lengths of loops, it is easy to make a distinction between 'infinite' string (winding around the box many times) and smaller loops, peaked at the minimum size of four lattice spacings. The analytic form of the small-loop distribution is well understood statistically. It is found that, for a cubic lattice, around $70 - 80\%$ of string exists as infinite string in this scenario, which has long been used as the generator of initial configurations for the numerical evolution of string networks [5–7].

If a non-minimal discretisation of the vacuum manifold is used (that is, in the case of a broken $U(1)$ symmetry, approximating S^1 with more than the smallest number of points, $\theta = 0,\ 2\pi/3,\ 4\pi/3$) and we employ a cubic lattice of points, in principle it is possible for all six faces of a fundamental cell to contain strings. Even in the minimal case, it is possible for four faces to do so. This requires a random choice to be made, pairing the incoming and outgoing strings. The only method that avoids this ambiguity is to use a tetrahedral lattice with a minimal discretisation, so that at most one string enters and leaves each cell. String configurations arising from this model have been analysed recently [8] and it is found that a slightly lower fraction (around 65%) exists as infinite string.

As a string network evolves, by means of intercommutation and expansion of the universe, it has been predicted and (to different extents) observed in the simulations that the characteristic lengths describing it approach a 'scaling regime', in which they grow in proportion to the horizon size. A typical evolving network will display an initial flurry of loop production before settling into this scaling regime with a few long strings and large loops per horizon volume continuing to (self-) intersect and produce smaller loops. As such, it has been supposed that the initial details of the string network are largely washed out after a few expansion times. This indeed seems to be the case, from both the numerical work and more recent analytic models of network evolution [9]. The question arises, though — is there a causal mechanism for creating a string distribution with significantly less infinite string? In the most extreme case, it is possible that if there were no infinite string at all, all the loops would disappear within a finite (and quite short) time. Recent work by Ferreira and Turok [10] partly confirms this, showing that a different type of scaling occurs in that case. One way of testing this idea is to attempt to rectify one of the major simplifications inherent in the Vachaspati-Vilenkin algorithm, and introduce a *distribution* of domain volumes in the initial conditions, instead of simply assuming that causally disconnected regions of one value of the field are of equal volume ($\sim \xi^3$).

II. IMPLEMENTING THE ALGORITHM

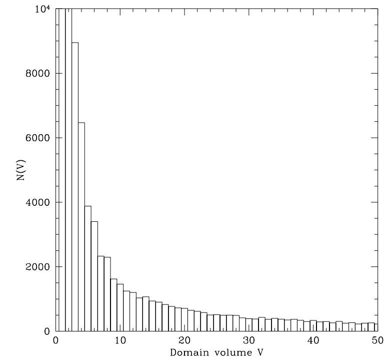
A cubic lattice was used with a near-continuous * representation of the vacuum manifold, despite the reduction in ambiguity that can be achieved with the tetrahedral lattice, as mentioned above, since it simplified the process

* i.e. a very high-density discretisation of the circle, that allows the use of integer arithmetic

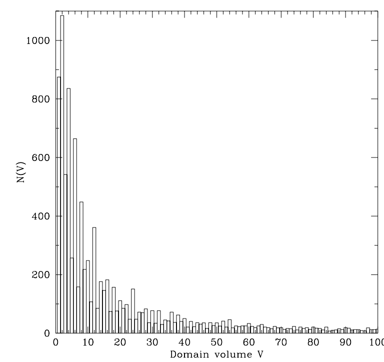
of creating a domain structure. Physical space is partitioned into regions of constant $U(1)$ phase by throwing down domains of random diameter within specified limits and gradually covering the lattice, dealing with the overlap and fragmentation of these regions as the box becomes filled with the broken phase. Roughly spherical domains were experimented with at first. However, after taking account of the significant domain overlap that resulted from the random filling of the lattice, and to make the task of ensuring that no domain was created entirely within another more straightforward, cubical domains were used. Once this was completed, strings were located and traced through the lattice, following the edges of either three or four adjacent domains.

It should be made clear that this is no more than a means of setting up a domain structure, and in no way claims to simulate the dynamics of an actual phase transition. Indeed, it is not obvious what order of transition the results of this algorithm apply to, though it would seem more closely related to string formation at the interfaces of expanding bubbles of the true vacuum, rather than the uniform emergence of a domain structure in a second-order transition. As a first guess we might expect a Gaussian distribution of domain volumes, peaked around some mean value. As it happens this is difficult to realise, and the size distribution appears to be more Poissonian (Fig. 1). The results are interesting nevertheless, and in particular the fact that the resulting form of the graph seems largely insensitive to modifications to the domain-laying algorithm.

The range of sizes of domains laid down was systematically increased in order to plot the fraction of the total string density as 'infinite' string, f_∞, against domain volume variance. The variance is normalised to the mean domain volume in order to remove effects due to uniform scaling-up of domain volumes. In the zero-variance limit the Vachaspati-Vilenkin result of $f_\infty \simeq 0.76$ is obtained. The precise value is weakly dependent on the imposed loop/'infinite' string cutoff — if we set the maximum size of a loop to be $4N$, $10N$ and $N^2/2$, where N is the length of the side of our box in units of the smallest possible domain size, the zero-variance values of f_∞ are 0.78, 0.77 and 0.75 respectively. Higher-variance values of f_∞ change by a similar amount. A cutoff of $N^2/4$ was used in the plots presented here.



(a)



(b)

FIG. 1. Histograms of domain volumes. Each illustrates range of domain sizes present for a particular realisation of the laying algorithm, which fills the box with domains of diameter randomly chosen in the range 1 to D. Figure (a) shows the results for a 100^3 box with D=5; (b) D=15.

III. RESULTS

In figure 2, each point is the average of 20 runs, with fixed limits on the range of sizes of domains laid down. The first point is the result of filling half the box with domains of side 1 or 2, randomly chosen. The remaining space is filled with unit domains, in order to achieve a low volume variance. Subsequent points correspond to the box being filled entirely with domains of sides between 1 and D (D=2,3,...,18).

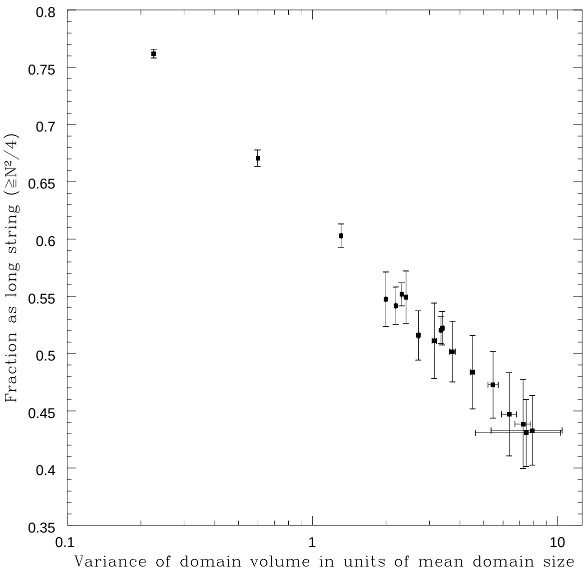


FIG. 2. Results for 100^3 lattice.

The initial decrease in f_∞ with increasing variance is perhaps intuitively understood. Typically, on a regular lattice, the string performs a self-avoiding random walk - one in which the string is not permitted to intersect itself or any other strings except at the 'origin'. This property is imposed on the strings by the restrictions of the lattice method. Such a walk has the property that the end-to-end distance l, in units of the step length, is related to the mean displacement R by

$$R \sim l^{3/5}.$$

In fact, the presence of other strings provides an extra repulsive effect and so gives the string near-Brownian characteristics ($R \sim l^{1/2}$). This is confirmed in the simulations — typical figures for the l exponent were $\simeq 0.47 \pm 0.04$ at zero variance. However, the presence of extended regions of space from which the string is excluded, i.e. larger domains, provides restrictions on the ability of the string to 'fold in' on itself. Effectively, in the region of these larger domains, we expect that a loop of a given radius will have a smaller perimeter than a similar loop in a region of unit domains. This will increase the density of loops below the cutoff size.

It is worth noting that as the variance increases, there exist more ways to fill the box and so a wider range of possible domain configurations. This also emphasises the point that volume variance is almost certainly not the only parameter describing the spatial distribution of phases that determines f_∞. Statistics at high values of N became unreliable, but it is intriguing to speculate whether further increases in the variance could well force all string to be in the form of small loops. The finite size of the simulation limits the maximum value of N we can reasonably investigate.

IV. PERCOLATION EFFECTS

Another way to observe a reduction in the density of infinite string is to impose a 'tilt' on the vacuum manifold, statistically favouring the occurrence of one phase [11].

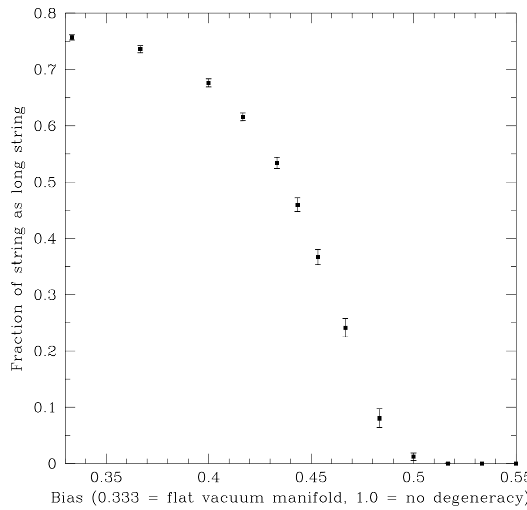


FIG. 3. Plot of f_∞ against the bias parameter γ for a zero-variance, 100^3 box. Each point is averaged over 20 realisations.

If we employ a three-point discretisation and gradually increase a bias parameter γ, such that $P(\text{phase 1}) = \gamma, P(2) = P(3) = \frac{1}{2}(1-\gamma)$, we find that f_∞ drops smoothly, reaching zero at $\gamma \simeq 0.5$ (Fig. 3). A phase is said to percolate when it is possible to trace continuous 'infinite' paths through that phase in the box.

Clearly there *is* a relation between the strings percolating (passing through the Hagedorn transition) and the percolation of phases in the box in this minimally discretised case. For a string to exist it must have all three phases around it. An infinite string will therefore ensure that all phases percolate (including diagonally adjacent regions). At least one phase percolates for all values of γ — the critical probability for the occurrence of one phase p_c, above which it percolates, is 0.31 [3]. In fact, we note that when $\gamma \simeq 0.5$, $P(2) = P(3) \simeq 0.25$, which is very near the percolation threshold.

It is interesting to ask whether there is a connection between increasing the variance of the domain volume and moving away from string percolation. Statistical fluctuations in the volume of the domains will result in the fractions of box volume occupied by each phase departing from 1/3, becoming more divergent as the range of sizes of domains increases. We suggest that this can be interpreted as an effective tilt of the vacuum manifold.

The fact that a bias will reduce the amount of long string is easily understood. We consider the probability p that a given plaquette is pierced by a string ($p = 8/27$ (≈ 0.30) in the case of three-point discretisation). Given

that we have an ingoing string through one face, what is the probability that this string will turn through $\pi/2$ in the cell under consideration? Obviously the opposite face has four independent phases (1,2 or 3) attached to it, so the probability of it containing an *outgoing* string is $p/2$. Given this configuration, the probability of the cell containing a further ingoing/outgoing string is 1/4. Thus the probability that our string continues through the cell undeviated, assuming we pair strings within the box randomly, is

$$\left(\frac{p}{2} \times \frac{3}{4}\right) + \left(\frac{p}{2} \times \frac{1}{4}\right) \times \frac{1}{2} = \frac{7}{16}p.$$

As we increase the bias parameter γ we obviously decrease the probability p. In fact,

$$p = 2\gamma(1-\gamma)^2,$$

giving values for p of 0.30, 0.15 and 0.02 for $\gamma = 1/3$, $2/3$ and $8/9$ respectively. Thus, strings will be more likely to fold up as γ grows, and the population of small loops will increase. This agrees with ref. [8], who point out that as the bias increases and the strings stop percolating, the fractal dimension of the strings becomes higher than two and they tend to 'crumple up' more — they become self-seeking random walks. Unfortunately, statistics were too poor to investigate any change in the fractal dimension of the strings as the bias or the variance was increased.

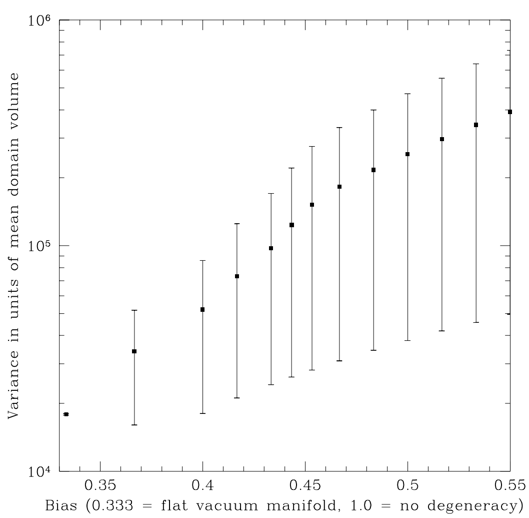


FIG. 4. Correlating the volume variance with the bias parameter γ

However, it is possible to investigate qualitatively the connection between domain volume variance and a vacuum tilt by calculating values for the variance for each value of the bias γ in figure 3. We set up the box by throwing down phases in single cells according to the biased probability distribution. We then group adjoining cells containing the same phase to form larger domains, whose volumes we calculate. The results are plotted in figure 4. The errorbars are misleading since there is clearly a correlation, and this is to be expected intuitively — the more one phase appears at the expense of others, the bigger the range of sizes of domains present.

Exploring the idea further, we calculate the bias parameter 'geometrically', given the volume occupied by each of the three phases in the box. The results of this procedure are shown in figure 5. The values of γ_{eff} are too low to correspond to those in the original figure (3), and the plots are not similar in form. However, the increase of γ_{eff} as f_∞ decreases is in agreement.

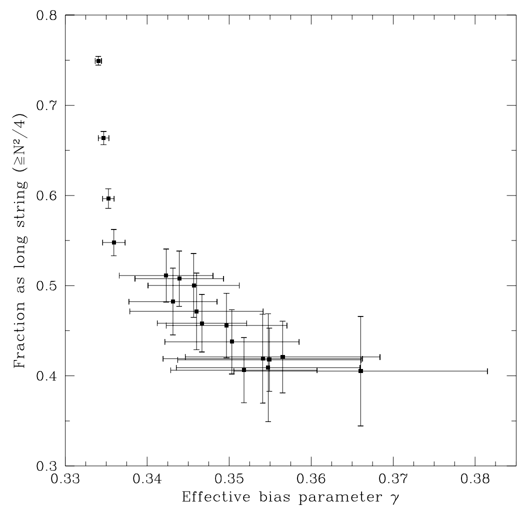


FIG. 5. Calculation of the effective bias parameter in the minimally-discretised case. For increasing values of N, a value for γ was calculated from the fractions of the box occupied by each of the three phases.

V. CONCLUSIONS

We have seen that a simple extension to the accepted numerical model for string formation can yield a significantly different estimate of the amount of infinite string present. It seems feasible that increasing the variance of the volumes of regions with different VEVs is equivalent to an effective tilt of the vacuum manifold, that leads to a reduction in the density of infinite string when considering a finite volume with periodic boundary conditions. Whether this is the case in the infinite-volume limit is more debatable. With three-point discretisation, the variance becomes ill-defined in this regime, as all three phases percolate. However, in this limit it is also unclear whether there is truly a population of infinite string, distinct from the $l^{-5/2}$ loop distribution, or if it is purely an artefact of the boundary conditions.

As yet there is no physical argument to suggest what the volume variance in a given phase transition will be — and even, considering the effects of phase equilibration at domain boundaries, how well-defined this quantity is. Models of dynamic defect formation, even with simplified treatments of the physics involved in a real phase

transition, may give improved predictions of the defect configurations [12].

One of the consequences of the existence of GUT-scale strings is the possibility of their being responsible for structure formation. It is only the infinite string and large loops that will survive long enough to be useful in this scenario, as a huge number of Hubble times elapse between string formation and when perturbations on interesting (galactic) scales will begin to grow. It may be that if the amount of long string present is very low, then their structure-seeding properties will be less significant than previously thought. Certainly, the proposed existence of a unique scaling solution for the string network, independent of initial conditions, would be put into doubt.

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[1] T.W.B.Kibble, *J. Phys. A* **9** (1976) 1387
[2] For a review see, for example, M.B. Hindmarsh, T.W.B.Kibble, *Rep. Prog. Phys.* **58** (1995) 477
[3] T. Vachaspati, A. Vilenkin, *Phys. Rev. D* **30** (1984) 2036
[4] D. Austin, E.J. Copeland, R.J.Rivers, *Phys. Rev. D* **49** (1994) 4089
[5] B. Allen, E.P.S. Shellard, *Phys. Rev. Lett.* **64** (1990) 119
[6] A. Albrecht, N. Turok, *Phys. Rev. D* **40** (1989) 973
[7] D.P. Bennet, F.R. Bouchet, *Phys. Rev. D* **41** (1990) 2408
[8] M.B. Hindmarsh, K. Strobl, Nucl. Phys. **B437** (1995) 471
[9] D. Austin, E.J. Copeland, T.W.B. Kibble, *Phys. Rev. D*,**48** (1993) 5594
[10] P. Ferreira, private communication
[11] T. Vachaspati, *Phys. Rev. D* **44** (1991) 3723
[12] J. Borrill, in preparation